ABSORPTION LINE FORMATION AND THE CURVE OF GROWTH

1. Introduction

Much of our knowledge of the physical conditions of astrophysical systems relies on spectroscopic observations. In this chapter we are going to look in some detail at the absorption lines formed when a background source of light is seen through a foreground 'screen' of absorbing gas. This situation is common, for example, when observing stars in our Galaxy, whose spectra bear the imprint of the Galactic interstellar medium, and distant QSOs, whose light traverses huge distances through space on its journey to Earth-bound telescopes. Intervening gas, whether within or without galaxies, will leave its signature in the QSO spectrum in the form of absorption lines at lower redshift than the QSO emission redshift, i.e. $z_{\rm abs} \leq z_{\rm em}$. Our ultimate aim is to deduce from measurements of these absorption lines the column densities and velocity dispersions of the absorbing atoms and ions.

2. Column Density

Consider an arbitrary structure of gas—a 'cloud'—with number density of atoms and ions which is a function of position within the cloud: $n\ ({\rm cm}^{-3}) = n(x, y, z)$. A sight-line to a background star or QSO will probe a finite distance through the cloud for a length L. At each infinitesimal location along the line of sight, from l to $l + dl$, the gas density sampled $n(l)$, will be a function of the xyz coordinate of $l = l(x, y, z)$. The column density N is defined as the integral of the number density along the line of sight:

$$N = \int_0^L n(l)\, dl \tag{10.1}$$

In practice one has very little, if any, knowledge of $n(x, y, z)$. However, the quantity N can, under favourable conditions, be deduced from the strengths and profiles of absorption lines. If column densities can be measured for different ionisation stages of the same ion, e.g. Si^0, Si^+, Si^{++}, Si^{3+} from absorption lines of Si I, Si II, Si III, Si IV, the ionisation conditions of the gas can be inferred. In some physical environments, for example clouds where H is predominantly neutral, the ionisation structure of the gas can be simple, with most of the atoms/ions of a given element being concentrated in a dominant ionisation stage. In this circumstances, if absorption lines from different elements are available, the chemical composition of the absorber can be investigated.

For these reasons, the column density N is the important quantity which we aim to deduce from the analysis of absorption lines.

3. Optical Depth

Consider a plane parallel slab of gas of thickness L along a coordinate x, so that x varies from 0 to L. The intensity of light of wavelength λ incident upon the slab at $x = 0$ is $I_{\lambda,0}$. The incremental extinction of the intensity of light over an infinitesimal distance from x to $x + dx$ is

$$dI_\lambda = -a_\lambda n I_\lambda dx \tag{10.2}$$

where a_λ (which has the units of a cross-section, that is cm^2) is the line absorption coefficient describing the absorption of photons by bound-bound atomic transitions. Following the absorption to a higher energy level, the electron falls back to its original state (possibly via intermediate states) re-emitting a photon (or photons). However, since such photons are re-emitted in a random direction, they are effectively scattered out of the observed line of sight to the background source.

The optical depth at wavelength λ

$$\tau_\lambda = \int_0^L a_\lambda n \, dx \tag{10.3}$$

is the integrated absorption of I_λ emerging from the slab at L, so that we can re-write eq. 10.2 as

$$dI_\lambda = -I_\lambda d\tau_\lambda \tag{10.4}$$

which has the solution

$$I_\lambda = I_{\lambda,0} \, e^{-\tau_\lambda} \tag{10.5}$$

Referring to 10.1 and 10.3 it can also be seen that

$$\tau_\lambda = N \cdot a_\lambda \tag{10.6}$$

4. Line Equivalent Width

The Equivalent Width W_λ of an absorption line is defined as the width, in wavelength units, of a rectangular strip of spectrum having the same area as the absorption line. That is, it is the width which the line would have, for the same energy taken out, if the intensity of the line were zero everywhere.

$$W_\lambda = \int_{-\infty}^{\infty} \frac{I_{\lambda,0} - I_\lambda}{I_{\lambda,0}} \, d\lambda \; = \; \int_{-\infty}^{\infty} \left(1 - e^{-\tau_\lambda}\right) \, d\lambda \tag{10.7}$$

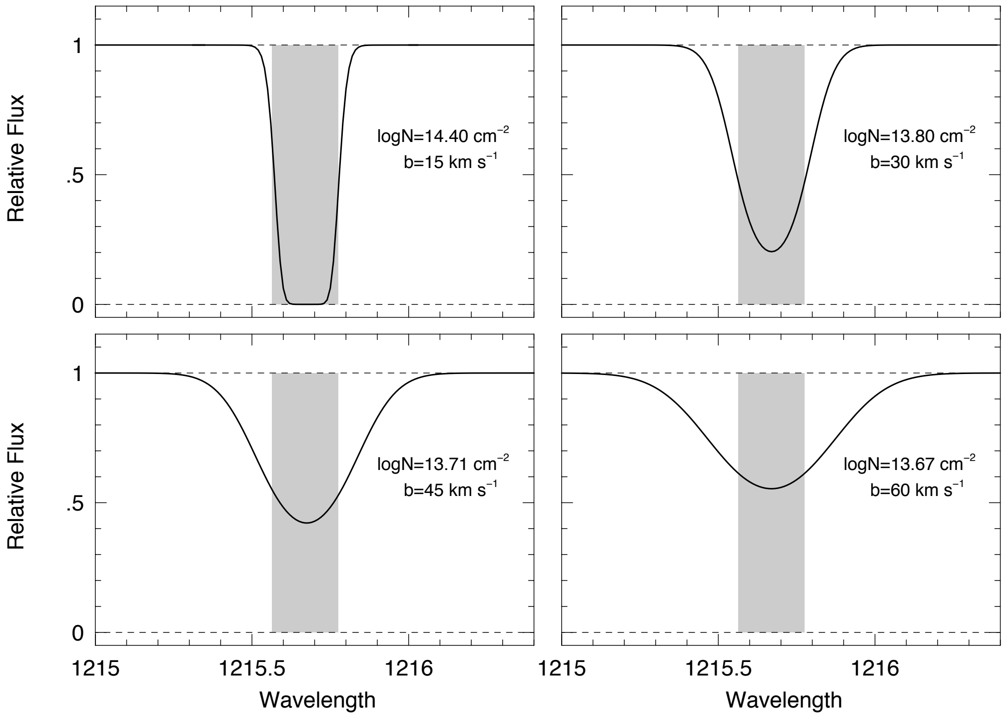

Figure 10.1: Four Lyα lines with the same equivalent width (shaded grey area) but different widths (and column densities). Figure courtesy of Chris Churchill.

The equivalent width is a **fudge**, made necessary by the fact that the resolution of spectrographs is often too coarse to resolve the intrinsic profiles of the absorption lines. What all instrument record is the *convolution* of the intrinsic line shape with the instrumental broadening function; the latter is often broader than the intrinsic width of the absorption line recorded. In these circumstances, we lose most of the information encoded in the line profile. However, the equivalent width is invariant to the convolution, and is thus a conserved quantity (modulo complications introduced by limited signal-to-noise ratio of the spectra). Under favourable conditions we can still recover the desired column density of the absorbers N from measurements of the equivalent width W_λ of an absorption line.

5. The Broadening Function

The line absorption coefficient at wavelength λ can be written as:

$$a_\lambda = a_0 \cdot \Phi_\lambda \tag{10.8}$$

where a_0 includes the atomic parameters of the transition

$$a_0 = \frac{\lambda^4}{8\pi c} \frac{g_k}{g_1} a_{k1} \tag{10.9}$$

where g_k and g_1 are the statistical weights of the excited and ground energy levels, a_{k1} is the transition probability and the other symbols have their usual meaning.[1]

Φ_λ is the broadening function, defined so that if an absorption does take place in the line, $\Phi_\lambda d\lambda$ is the probability that the wavelength of the absorbed photon lies between λ and $\lambda + d\lambda$. Thus:

$$\int_{-\infty}^{+\infty} \Phi_\lambda \, d\lambda = 1 \tag{10.10}$$

The value of Φ_λ is large near λ_0 (the line centre) and falls off rapidly at longer and shorter wavelengths.

For interstellar conditions we need to consider only two broadening processes:

1. Natural broadening, due to the intrinsic uncertainty ΔE in the energy of the upper atomic level k, and
2. Doppler broadening due to the motions of the absorbers.

[1] In the interstellar medium the densities of matter and radiation are normally sufficiently low that most atoms and ions are predominantly in their ground electronic state. We can thus ignore processes such as stimulated emission and pressure broadening, both of which do need to be taken into account in the theory of *stellar* line formation, for example.

For an interstellar atom at rest:

$$\phi_\lambda(v=0) = \frac{1}{\pi}\frac{\delta_k}{\delta_k^2 + (\lambda-\lambda_0)^2} \tag{10.11}$$

δ_k is the radiation damping constant in wavelength units:

$$\delta_k = \frac{\lambda^2}{4\pi c}\sum_{E_r<E_k} a_{kr} \tag{10.12}$$

If the atoms are not at rest, but have a distribution of radial velocities along the line of sight $\Psi(v)$, then:

$$\Phi_\lambda = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\delta_k}{\delta_k^2 + \left[\lambda - \lambda_0\left(1+\frac{v}{c}\right)\right]^2}\Psi(v)dv \tag{10.13}$$

For a single ‘cloud’ with a Maxwellian distribution of velocities:

$$\Psi(v) = \frac{1}{\sqrt{\pi}b}\exp\left[-\frac{(v-v_0)^2}{b^2}\right] \tag{10.14}$$

where b is the Doppler width, corresponding to the most probable speed along the line of sight. For thermal motions alone:

$$b_{\rm th} = \left(\frac{2kT}{m}\right)^{1/2} \tag{10.15}$$

More generally, b has components from both thermal and bulk motions:

$$b^2 = b_{\rm th}^2 + b_{\rm turb}^2 \tag{10.16}$$

If M 'clouds' are intersected along the line of sight:

$$\Psi(v) = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{M} \frac{p_i}{b_i} \exp\left[-\frac{(v - v_{0,i})^2}{b_i^2}\right] \qquad (10.17)$$

Both natural and Doppler broadening need to be taken into account. Generally, the damping constant δ_k is much smaller than the Doppler constant b (both in wavelength units):

$$\beta = \frac{\delta_k}{b} \leq 0.005 \qquad (10.18)$$

However, comparing the expressions for ϕ_λ and $\Psi(v)$ (eqs. 10.11 and 10.14), it can be seen that the probability that an atom will absorb away from λ_0 falls off:

- as the inverse square of $\Delta\lambda = (\lambda - \lambda_0)$ for natural broadening, and
- exponentially with $\Delta\lambda = (\lambda - \lambda_0)$ for Doppler broadening.

Equating 10.11 and 10.14 we find that at $\Delta\lambda(\cdot\frac{c}{\lambda}) \simeq 3b$ the probabilities of absorption due to natural and Doppler broadening are comparable (*for the values of a_{kr} in 10.12 appropriate to the most commonly seen interstellar absorption lines*). For $\Delta\lambda < 3b$ atoms absorb primarily because of their motion; for $\Delta\lambda > 3b$ they absorb primarily because of natural damping.

6. The Curve of Growth

6.1 Overview

The full expression for the line optical depth at wavelength λ is given (10.6, 10.8, 10.11, 10.14) by:

$$\tau_\lambda = N\, a_0\, \phi_\lambda \otimes \Psi(v) \tag{10.19}$$

where the convolution of the natural broadening and Doppler broadening functions is as given in eq. 10.13. The expression 10.19 for τ_λ is often referred to as the Voigt function.

Integration of 10.7 then gives the sought-after relationship between the Equivalent Width W_λ of an absorption line and the column density N of absorbing atoms. This relationship, which is illustrated in Fig. 10.2, is known as the Curve of Growth, because it describes how W_λ grows with increasing N.

The precise functional dependence of W_λ on N is sensitive to the optical depth at the line core, τ_0. Qualitatively, as τ_0 increases from $\tau_0 \ll 1$, the residual intensity I_λ in the line core decreases (the line depth increases) until all the photons at the line core are removed from the incoming beam. At this point, the absorption line is considered to be 'saturated'. As τ_0 increases further, very little additional light is removed from the beam until we reach a regime where the optical depth becomes significant at wavelengths far from the line centre, where absorption takes place from the natural broadening of the line. The equivalent width can then grow again, as the line develops characteristic 'damping wings' [where the absorption coefficient falls off as the inverse square of $(\lambda - \lambda_0)$].

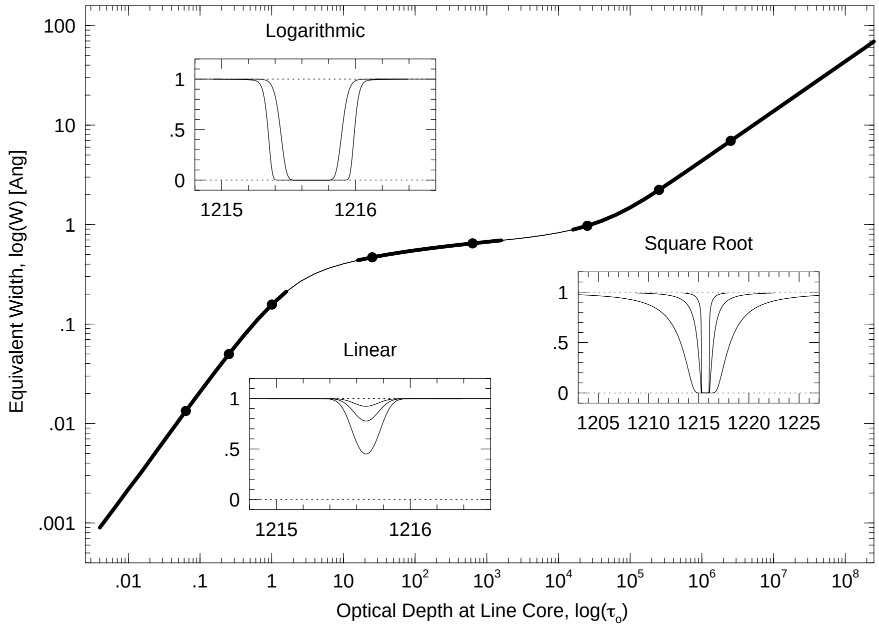

Figure 10.2: Example of a curve of growth for a Lyα line of H^0 with $b = 30\,\mathrm{km\ s^{-1}}$. The three regimes discussed in the text, the linear, flat, and damping part of the COG are shown by thicker curves. Corresponding line absorption profiles are shown for each regime and their locations on the COG are marked with filled dots. The wavelength (x-axis) scale in the panel for lines on the damping part of the COG has been expanded relative to the other two panels to illustrate the large extent of damping wings. (Figure courtesy of Chris Churchill).

6.2 Measuring Column Densities

This behaviour defines three distinct portions of the Curve of Growth, illustrated in Fig. 10.2:

1. **The linear part**, where $\tau_0 < 1$ and $W_\lambda \propto N$. The absorption line is optically thin and W_λ is a sensitive measure of N, irrespectively of the value of the Doppler parameter b.

2. **The flat, or logarithmic, part**, where $10 \lesssim \tau_0 \lesssim 10^3$ and $W_\lambda \propto b\sqrt{\ln(N/b)}$. The absorption line is optically thick and W_λ is *not* a good measure of N, but is sensitive to the Doppler parameter b.

3. **The damping, or square root, part**, where $\tau_0 \gtrsim 10^4$ and $W_\lambda \propto \sqrt{N}$. In this regime, the optical depth in the damping wings provides an accurate estimate of N.

Using the approximation $e^{-x} \simeq 1 - x$ when $x \ll 1$, it can be readily seen from eq. 10.7 that on the linear part of the COG:

$$W_\lambda = \int_{-\infty}^{\infty} \tau_\lambda \, d\lambda \;=\; N\, a_0 \int_{-\infty}^{\infty} \Phi_\lambda \, d\lambda \;=\; N\, a_0 \qquad (10.20)$$

since we defined $\int_{-\infty}^{\infty} \Phi_\lambda \, d\lambda = 1$ (eq. 10.10).

Entering the numerical values, eq. 10.20 reduces to:

$$N = 1.13 \times 10^{20} \cdot \frac{W_\lambda}{\lambda^2 f} \;\; \mathrm{cm}^{-2} \qquad (10.21)$$

where λ and W_λ are both in Å and f is the 'oscillator strength' (available from compilations of atomic data), related to g_1, g_k and a_{k1} by:

$$\lambda\, g_1 f_{1k} = \frac{m_e c\, \lambda^3}{8\pi^2 e^2} \cdot g_k a_{k1} \qquad (10.22)$$

For the Lyman α line of neutral hydrogen, we have $\lambda = 1215.6701$ Å and $f = 0.4164$; thus

$$N(\mathrm{H\ I}) = 1.84 \times 10^{14} \cdot \mathrm{W}_\lambda \;\; \mathrm{cm}^{-2} \qquad (10.23)$$

(with W_λ measured in Å). Therefore a Lyman α absorption line with an equivalent width (in the rest frame, of course!) W_λ = 100 mÅ corresponds to neutral hydrogen column density N(H I)= $2 \times 10^{13}\,\mathrm{cm}^{-2}$.

On the damping part of the COG:

$$W_\lambda = 2\,(N a_0\, \delta_k)^{1/2} \;\;\; \text{Å} \qquad (10.24)$$

where N is in cm^{-2}. For the Lyman α line:

$$N(\mathrm{H\ I}) = 1.88 \times 10^{18} \cdot W_\lambda^2 \ \ \mathrm{cm}^{-2} \tag{10.25}$$

(with W_λ measured in Å), so that N(H I)$= 2 \times 10^{20}$ cm^{-2} corresponds to $W_\lambda = 10$ Å. (In practice, however, it is more common to use the profile of the damping wings to deduce N(H I), because it becomes impractical to measure the equivalent width of an absorption line with extended damping wings.)

6.3 Where on the Curve of Growth?

In general, it is not possible to know from the equivalent width alone where a single absorption line falls on the COG, unless the line profile is adequately resolved (i.e. unless the intrinsic shape of the line has not been smeared out by the spectrograph used to record it). For a single, unresolved line the relationship between column density N and equivalent width W_λ is degenerate: the same value of W_λ can imply significantly smaller/larger values of N for larger/smaller values of the velocity dispersion parameter b.

However, if several transitions with different values of $f\lambda$ originating from the same atomic level are available, one can construct an 'empirical' curve of growth and deduce both N and b (see Figures 10.3 and 10.4).

Many of the most commonly observed absorption lines are doublets, that is transitions from the ground state to an excited state consisting of two closely spaced sublevels (L-S coupling). The ratio of the equivalent widths of such pairs of lines, the doublet ratio, can be a useful pointer to the region of COG where the lines fall. For example, for two transitions whose values of g_k in eq. 10.9 differ by a factor of two, e.g. C IV $\lambda\lambda 1548, 1550$, the doublet ratio is:

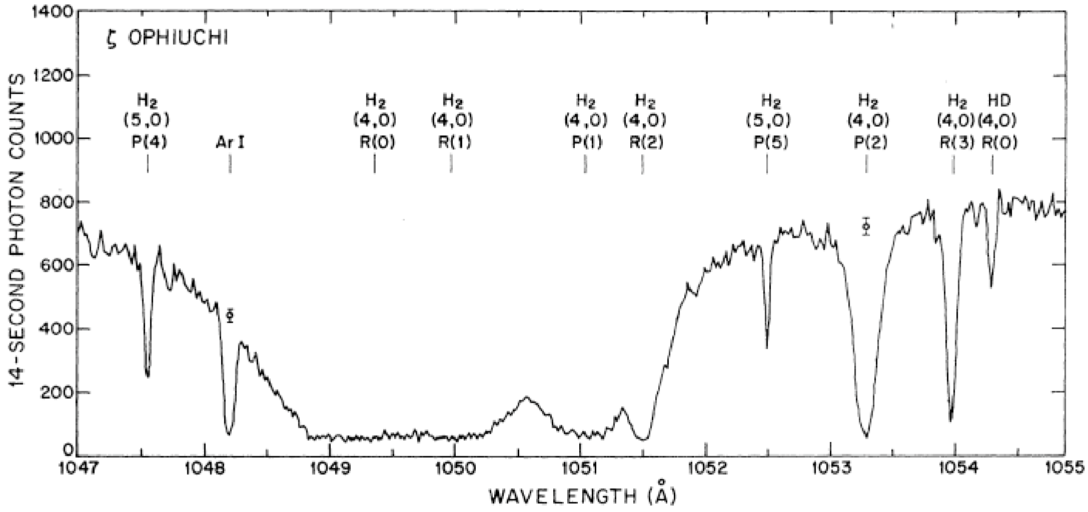

Figure 10.3: Portion of the far-ultraviolet spectrum of the bright star ζ Ophiuchi obtained in the early 1970s with the *Copernicus* satellite. This portion includes several absorption lines of molecular hydrogen, H_2, of widely different strengths, from unsaturated lines on the linear part of the curve of growth, to lines exhibiting clear damping wings.

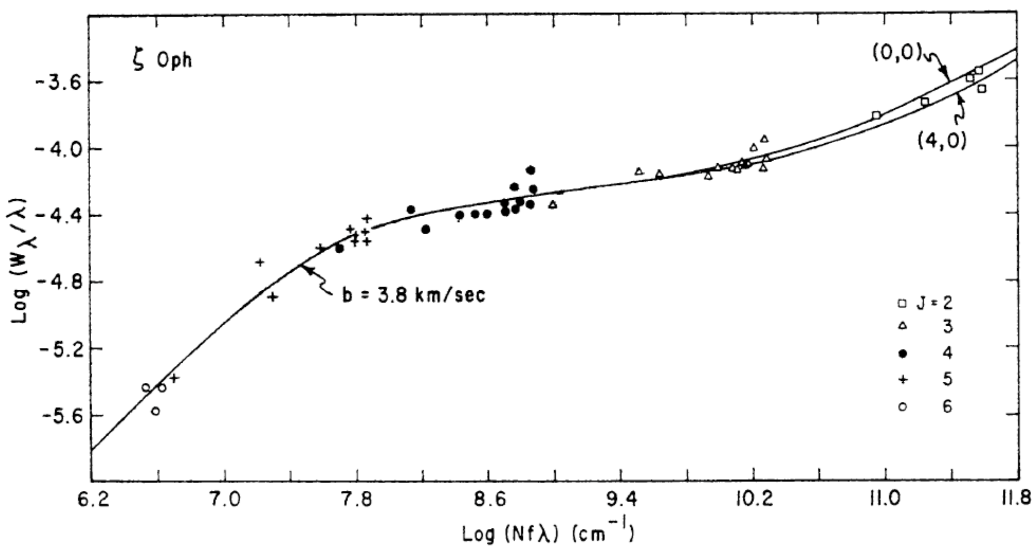

Figure 10.4: Empirical curve of growth defined by the many H_2 lines detected in the spectrum of ζ Oph. Within the errors, the absorption lines fit a 'single cloud' curve of growth with a Doppler parameter $b = 3.8\,\mathrm{km\ s^{-1}}$.

- $W_{\lambda 1548}/W_{\lambda 1550} = 2.0$ on the linear part of the COG.
- $W_{\lambda 1548}/W_{\lambda 1550} = 1.1$ on the flat part of the COG
- $W_{\lambda 1548}/W_{\lambda 1550} = \sqrt{2}$ on the damping part of the COG.

Therefore, if one measures $W_{\lambda 1548}/W_{\lambda 1550} > 1.4$, one can safely conclude that the lines are on the linear part of the curve of growth, and use their W_λ to deduce N(C IV).

In H I regions of the interstellar medium, absorption lines of the dominant ion stage of element X are close to the linear part of the COG ($\tau_0 < 2$) if the following conditions are satisfied:

- $b > 3$ km s^{-1}
- N(H I)$< 1 \times 10^{21}$ cm^{-2}
- $A + \log(f\lambda) < -5.4$

where A is the log of the abundance of element X relative to H.

7. Dispensing of the Equivalent Width Altogether

In some cases the profiles of the absorption lines are resolved. This happens when the resolving power of the spectrograph used to record the lines is comparable to, or greater than, the intrinsic width of the lines.[2] For interstellar absorption lines in the solar vicinity, observed against the spectra of bright O and B stars, this was achieved 35 years ago (L. Hobbs and collaborators). For QSO absorption lines, this only became possible in the mid-90s with the advent of echelle spectrographs on large (8-10 m) telescopes, particularly HIRES on Keck and UVES on VLT.

If the lines are resolved, we can dispense altogether of the notion of equivalent width, since we can measure the optical depth directly at each velocity (or wavelength) interval over which absorption takes place. Integration of the optical depth over the absorption line profile (the 'optical depth method') then leads to the column density:

$$N(v) = 3.77 \times 10^{14} \cdot \frac{\tau(v)}{f\lambda} \ \ \mathrm{cm}^{-2} \left(\mathrm{km\ s}^{-1}\right)^{-1} \tag{10.26}$$

For the Lyman α line:

$$N(v) = 7.45 \times 10^{11} \cdot \tau(v) \ \ \mathrm{cm}^{-2} \left(\mathrm{km\ s}^{-1}\right)^{-1} \tag{10.27}$$

and

$$N(\mathrm{H\ I})_{\mathrm{tot}} = \sum_{1}^{n} N(v) \cdot d(v) \tag{10.28}$$

where the summation is over the n velocity intervals, each of width $d(v)$, spanned by the absorption line.

[2]Typical b values of intergalactic Lyα absorption lines are $b \simeq 20 - 30\,\mathrm{km\ s}^{-1}$. For a Gaussian distribution of velocities, these b values correspond to Full Widths at Half Maximum, FWHM $= 1.665\,b \simeq 35 - 50\,\mathrm{km\ s}^{-1}$. The resolving power of a spectrograph is usually expressed as $R = \lambda/\Delta\lambda = c/\Delta v$, where $\Delta\lambda$ and Δv are the Full Widths at Half Maximum—respectively in wavelength and velocity units—of the instrumental broadening function.

8. The Column Density Distribution

Using the methods outlined above, it is possible to analyse the entire Lyα forest and deduce the column density of neutral hydrogen, $N_{\rm H\,I}$, of individual 'clouds' (although it is somewhat arbitrary what you call a cloud).

When we consider the distribution of values of $N_{\rm H\,I}$, we find that to a first approximation a single power law of the form:

$$f(N_{\rm H\,I}) = B \cdot N_{\rm H\,I}^{-\beta} \qquad (10.29)$$

where $f(N_{\rm H\,I})$ is the number of Lyα lines with column densities between $N_{\rm H\,I}$ and $N_{\rm H\,I} + dN_{\rm H\,I}$, fits all Ly$\alpha$ lines over 10 orders of magnitude in $N_{\rm H\,I}$, from $\log N_{\rm H\,I} = 12$ to 22.

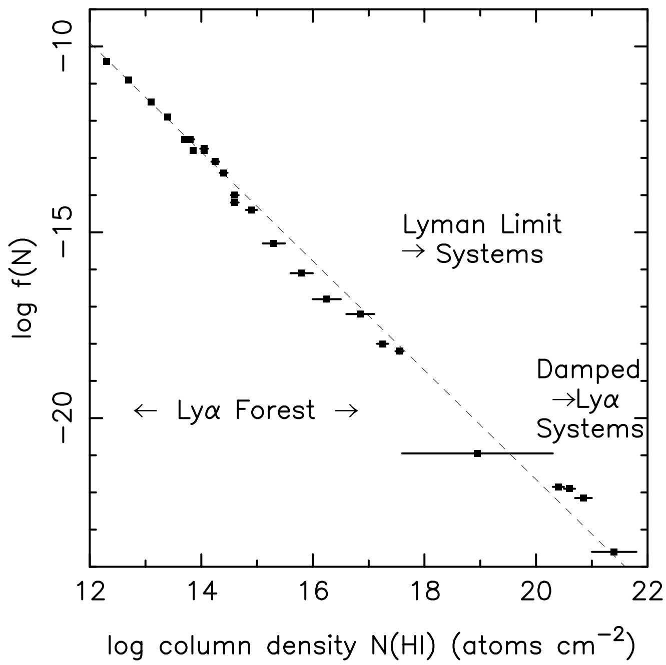

Figure 10.5: The observed distribution function of neutral hydrogen column densities in QSO absorbers (reproduced from Storrie-Lombardi & Wolfe 2000).

At $z \simeq 3$, the best fitting values are $\beta = 1.46$ and $B = 5.3\times10^7$ (for $\Omega_{\rm M} = 0.3, \Omega_\Lambda = 0.7$). The column density distribution function is an important property of the Lyα forest and QSO absorbers in general, as we shall see later on in the course.